3.22 Poisson Distribution

Poisson Process

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Properties
Poisson Process

Let $N(t)$ be a **counting process**. That is, $N(t)$ is the number of occurrences (or arrivals, or events) of some process over the time interval $[0, t]$. $N(t)$ looks like a step function.

Examples: $N(t)$ could be any of the following.
(a) Cars entering a shopping center (time).
(b) Defects on a wire (length).
(c) Raisins in cookie dough (volume).
Let $\lambda > 0$ be the average number of occurrences per unit time (or length or volume).

In the above examples, we might have:

(a) $\lambda = 10/\text{min.}$  
(b) $\lambda = 0.5/\text{ft.}$  
(c) $\lambda = 4/\text{in}^3$.

A Poisson process is a specific counting process. . .

First, some notation: $o(h)$ is a generic function that goes to zero faster than $h$ goes to zero.
Definition: A **Poisson process** is one that satisfies the following assumptions:

(1) There is a short enough interval of time, say of length \( h \), such that, for all \( t \),

\[
\Pr(N(t + h) - N(t) = 0) = 1 - \lambda h + o(h)
\]
\[
\Pr(N(t + h) - N(t) = 1) = \lambda h + o(h)
\]
\[
\Pr(N(t + h) - N(t) \geq 2) = o(h)
\]

(2) If \( t_1 < t_2 < t_3 < t_4 \), then \( N(t_4) - N(t_3) \) and \( N(t_2) - N(t_1) \) are **indep** RV's.
(1) Arrivals basically occur one-at-a-time, and then at rate $\lambda$/unit time. (We must make sure that $\lambda$ doesn’t change over time.)

(2) The numbers of arrivals in two disjoint time intervals are indep.
Poisson Process Example: Neutrinos hit a detector. Occurrences are rare enough so that they really do happen one-at-a-time. You never get arrivals of groups of neutrinos. Further, the rate doesn’t vary over time, and all arrivals are indep of each other.

Anti-Example: Customers arrive at a restaurant. They show up in groups, not one-at-a-time. The rate varies over the day (more at dinnertime). Arrivals may not be indep. This ain’t a Poisson process.
Poisson Distribution

Definition: Let $X$ be the number of occurrences in a Poisson($\lambda$) process in a *unit interval* of time. Then $X$ has the **Poisson distribution** with parameter $\lambda$.

Notation: $X \sim \text{Pois}(\lambda)$.

Theorem/Definition: $X \sim \text{Pois}(\lambda) \Rightarrow$

$$\Pr(X = k) = e^{-\lambda} \frac{\lambda^k}{k!}, \quad k = 0, 1, 2, \ldots$$
Remark: The value of \( \lambda \) can be changed simply by changing the units of time.

Example:

\( X = \# \) calls to a switchboard in 1 minute \( \sim \) Pois(3) 
\( Y = \# \) calls to a switchboard in 5 minutes \( \sim \) Pois(15) 
\( Z = \# \) calls to a switchboard in 10 sec \( \sim \) Pois(0.5)
Properties

Theorem: $X \sim \text{Pois}(\lambda) \Rightarrow \text{mgf is } M_X(t) = e^{\lambda(e^t-1)}$.

Proof:

\[
M_X(t) = E[e^{tX}] = \sum_{k=0}^{\infty} e^{tk} \left( \frac{e^{-\lambda} \lambda^k}{k!} \right)
\]

\[
= e^{-\lambda} \sum_{k=0}^{\infty} \frac{(\lambda e^t)^k}{k!}
\]

\[
= e^{-\lambda} e^{\lambda e^t}.
\]
Theorem: \( X \sim \text{Pois}(\lambda) \Rightarrow \mathbb{E}[X] = \text{Var}(X) = \lambda. \)

Proof (using mgf):

\[
\mathbb{E}[X] = \frac{d}{dt} M_X(t) \bigg|_{t=0} \\
= \frac{d}{dt} e^{\lambda(e^t-1)} \bigg|_{t=0} \\
= \lambda e^t M_X(t) \bigg|_{t=0} \quad \text{(chain rule)} \\
= \lambda \quad \text{(after algebra)}.
\]
Similarly,

\[ E[X^2] = \frac{d^2}{dt^2} M_X(t) \bigg|_{t=0} = \frac{d}{dt} \left( \frac{d}{dt} M_X(t) \right) \bigg|_{t=0} \]

\[ = \lambda \frac{d}{dt} \left( e^t M_X(t) \right) \bigg|_{t=0} \]

\[ = \lambda \left[ e^t M_X(t) + e^t \frac{d}{dt} M_X(t) \right] \bigg|_{t=0} \]

\[ = \lambda e^t \left[ M_X(t) + \lambda e^t M_X(t) \right] \bigg|_{t=0} \]

\[ = \lambda (1 + \lambda). \]
Thus,

$$\text{Var}(X) = \mathbb{E}[X^2] - (\mathbb{E}[X])^2 = \lambda(1 + \lambda) - \lambda^2 = \lambda.$$ 

Done.

Example: Calls to a switchboard arrive as a Poisson process with rate 3 calls/min.

Let $X =$ number of calls in 40 sec. So $X \sim \text{Pois}(2)$.

$$\mathbb{E}[X] = \text{Var}(X) = 2, \quad \Pr(X \leq 3) = \sum_{k=0}^{3} e^{-2} \frac{2^k}{k!}$$
Theorem (Additive Property of Poissons): Suppose $X_1, \ldots, X_n$ are *indep* with $X_i \sim \text{Pois}(\lambda_i), \ i = 1, \ldots, n$. Then

$$Y \equiv \sum_{i=1}^{n} X_i \sim \text{Pois}(\sum_{i=1}^{n} \lambda_i).$$

Proof:

$$M_Y(t) = \prod_{i=1}^{n} M_{X_i}(t) \quad (X_i \text{'s indep})$$

$$= \prod_{i=1}^{n} e^{\lambda_i(e^t-1)} = e^{(\sum_{i=1}^{n} \lambda_i)(e^t-1)},$$

which is the mgf of the $\text{Pois}(\sum_{i=1}^{n} \lambda_i)$ distribution.